

OPTIMIZATION OF ONE MARKETING RELATION MODEL WITH DELAY

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Abstract. The nonlinear dynamical model for one class of marketing relation with delay in control is given. For a corresponding optimization problem the existence theorem of optimal control and necessary optimality conditions are provided. In the linear case found all controls which are doubtful on optimality. An example is considered.

Keywords: Market model, optimization, existence theorem, necessary optimality conditions.

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1 Introduction. Mathematical model

As is known the real controlled processes contain effects with delayed action and are described by differential equations with delay in control. A certain model of demand and supply is suggested in (Kharatishvili et al., 2004).

Let market relation is characterized by demand and supply functions $D(p)$ and $S(p)$, respectively. Here the function $p = p(t)$ is price of a good, changing over time. Suppose that at time t consumer demand will be satisfied with a preliminary order at time $t - \tau$, where $\tau > 0$ is so called delay parameter.

The function

$$R(t) = D(p(t)) - S(p(t - \tau))$$

we call the disbalance index.

If $R(t) = 0$ then at the moment t we do not have disbalance between supply and demand, and the customer will buy exactly the quantity of goods he needs.

It is clear, that at various time moment t the disbalance index $R(t)$ is possible to be not positive as well as positive. If $R(t) > 0$ at time t , then demand exaggerates supply, If $R(t) < 0$ then supply exaggerates demand. To describe development of marketing relation process in time, i.e. create dynamical model, we consider the integral index of disbalance

$$x(t) = R(t_0) + \int_{t_0}^t R(s)ds, t_0 \leq t \leq t_1. \quad (1)$$

The function $x(t)$ gives complete information about the disbalance from the initial time t_0 to any time t .

From (1) we get the differential equation with delay in control

$$\dot{x}(t) = D(p(t)) - S(p(t - \tau)),$$

$$x(t_0) = x_0 := R(t_0).$$

It is natural to assume that in the considered time interval the price function satisfied the condition

$$p(t) \in P = [\alpha, \beta], \beta > \alpha \geq 0, t \in [t_0, t_1].$$

2 Statement of the problem

Let $I = [t_0, t_1]$ be a given interval and $\tau > 0$ be a given number, with $t_0 \geq \tau, t_1 > t_0 + \tau$; let x_1 be a given number and let Ω be a set of measurable control functions $p(t) \in P$.

We note that, as rule, optimal control there exists in class of measurable functions. Therefore here is considered the class Ω .

Below, on the basis of necessary conditions will be shown that the optimal control there exists in class of the piecewise constant functions.

To each control $p(t) \in \Omega$ we assign the scalar delay differential equation

$$\dot{x}(t) = D(p(t)) - S(p(t - \tau)), t \in I \tag{2}$$

with the initial condition

$$x(t_0) = x_0, \tag{3}$$

where $D(p)$ and $S(p)$ are continuous functions on the set P .

Definition 1. Let $p(t) \in \Omega$. A scalar function $x(t) = x(t; p), t \in I$ is called a solution of equation (2) with the initial condition (3), or a solution corresponding to $p(t)$, if it satisfies condition (3), is absolutely continuous on the interval I and satisfies equation (2) almost everywhere (a. e.) on I .

Definition 2. A control $p(t) \in \Omega$ is said to be admissible if the corresponding solution $x(t)$ satisfies the condition

$$x(t_1) = x_1. \tag{4}$$

Denote by Ω_0 the set of admissible controls.

Definition 3. A control $p_0(t) \in \Omega_0$ is said to be optimal if for an arbitrary $p(t) \in \Omega_0$ the inequality

$$J(p_0) \leq J(p) \tag{5}$$

holds. Here

$$J(p) = \int_{t_0}^{t_1} p(t) dt.$$

(2)-(5) is called the optimization problem with delay in control.

3 Existence theorem and necessary optimality conditions.

Example

Theorem 1. (Kharatishvili & Tadumadze, 2007, Theorem 9.3.1, p.164). There exists an optimal control $p_0(t) \in \Omega_0$ if the following conditions hold:

- a) $\Omega_0 \neq \emptyset$;
- b) the set

$$\left\{ \left(\begin{array}{c} p \\ D(p) \\ S(p) \end{array} \right) : p \in P \right\}$$

is convex.

Theorem 2. (Kharatishvili & Tadumadze, 2007, Theorem 6.3.1, p.99). Let $u_0(t) \in \Omega_0$ be an optimal control. There exists a nonzero vector $\psi = (\psi_0, \psi_1)$, with $\psi_0 \leq 0$, such that the following first order necessary conditions hold:

1) a. e. on $t \in [t_0 - \tau, t_0]$

$$\psi_1 S(p_0(t)) = \min_{p \in P} \psi_1 S(p) = m_0;$$

2) a. e. on $t \in [t_0, t_1 - \tau]$

$$\psi_0 p_0(t) + \psi_1 D(p_0(t)) - \psi_1 S(p_0(t)) = \max_{p \in P} [\psi_0 p + \psi_1 D(p) - \psi_1 S(p)] = m_1;$$

3) a. e. on $t \in [t_1 - \tau, t_1]$

$$\psi_0 p_0(t) + \psi_1 D(p_0(t)) = \max_{p \in P} [\psi_0 p + \psi_1 D(p)] = m_2.$$

Let us introduce the sets:

$$P_0 = \{p \in P : \psi_1 S(p) = m_0\}, P_1 = \{p \in P : \psi_0 p + \psi_1 D(p) - \psi_1 S(p) = m_1\},$$

and

$$P_2 = \{p \in P : \psi_0 p + \psi_1 D(p) = m_2\}.$$

Theorem 3. Let the sets $P_i, i = 0, 1, 2$ be finite. Then there exists an optimal control $p_0(t)$, that is the piecewise constant function.

Theorem 3 is a simply corollary of Theorems 1 and 2.

Remark 1. Let $\psi_1 = 0$ then $\psi_0 < 0$. From Theorem 2 it follows that $p_0(t) = \alpha, t \in I$ and $P_0 = P$. In this case the optimal control $p_0(t)$ on the interval $[t_0 - \tau, t_0]$ is so called singular (Mardanov et al., 2013).

Below we consider a particular case of the problem (2)-(5). Namely, let $D(p) = ap + b$ and $S(p) = cp + d$, where $a < 0$ and $c > 0$. Thus, we have the following optimization problem

$$\begin{cases} \dot{x}(t) = ap(t) - cp(t - \tau) + b - d, t \in I, p(t) \in \Omega, \\ x(t_0) = x_0, x(t_1) = x_1, \\ J(p) \rightarrow \min. \end{cases} \quad (6)$$

Remark 2. Let for problem (6) $\Omega_0 \neq \emptyset$. Then there exist an optimal control $p_0(t)$ (see Theorem 1).

Theorem 4. Let $p_0(t)$ be an optimal control for the optimization problem (6). There exists a nonzero vector $\psi = (\psi_0, \psi_1)$, with $\psi_0 \leq 0$, such that the following conditions hold:

4. a. e. on $t \in [t_0 - \tau, t_0]$

$$\psi_1 p_0(t) = \min_{p \in P} \psi_1 p;$$

5. a. e. on $t \in [t_0, t_1 - \tau]$

$$[\psi_0 + (a - c)\psi_1] p_0(t) = \max_{p \in P} [\psi_0 + (a - c)\psi_1] p;$$

6. a. e. on $t \in [t_1 - \tau, t_1]$

$$[\psi_0 + a\psi_1] p_0(t) = \max_{p \in P} [\psi_0 + a\psi_1] p.$$

Theorem 4 is a corollary of Theorem 2.

Remark 3. Let $\psi_1 \neq 0, \psi_0 + (a - c)\psi_1 \neq 0$ and $\psi_0 + a\psi_1 \neq 0$. Then $P_0 = P_1 = P_2 = \{\alpha, \beta\}$ and optimal control $p_0(t)$ is the piecewise constant function (see Theorem 4).

Below are given all possible cases for finding of the optimal control $p_0(t)$.

I. Let $\psi_1 = 0$ then $\psi_0 \neq 0$. On the interval $[t_0 - \tau, t_0]$ the control $p_0(t)$ is singular, i. e. $p_0(t) = p \in P$ and $p_0(t) = \alpha, t \in I$.

II. Let $\psi_1 > 0$ then $\psi_0 + (a - c)\psi_1 < 0$. Consequently, $p_0(t) = \alpha$ for $t \in [t_0 - \tau, t_1]$.

III. Let $\psi_1 < 0$ and $\psi_0 + (a - c)\psi_1 < 0, \psi_0 + a\psi_1 < 0$. Then,

$$p_0(t) = \begin{cases} \beta, t \in (t_0 - \tau, t_0), \\ \alpha, t \in I. \end{cases}$$

IV. Let $\psi_1 < 0$ and $\psi_0 + (a - c)\psi_1 < 0, \psi_0 + a\psi_1 > 0$. Then,

$$p_0(t) = \begin{cases} \beta, t \in (t_0 - \tau, t_0), \\ \alpha, t \in (t_0, t_1 - \tau), \\ \beta, t \in (t_1 - \tau, t_1). \end{cases}$$

V. Let $\psi_1 < 0$ and $\psi_0 + (a - c)\psi_1 > 0, \psi_0 + a\psi_1 < 0$. Then,

$$p_0(t) = \begin{cases} \beta, t \in (t_0 - \tau, t_1 - \tau), \\ \alpha, t \in (t_1 - \tau, t_1). \end{cases}$$

VI. Let $\psi_1 < 0$ and $\psi_0 + (a - c)\psi_1 > 0, \psi_0 + a\psi_1 > 0$. Then, $p_0(t) = \beta, t \in [t_0 - \tau, t_1]$.

VII. Let $\psi_1 < 0$ and $\psi_0 + (a - c)\psi_1 = 0$. Then, $\psi_0 + a\psi_1 < 0$. Thus,

$$p_0(t) = \begin{cases} \beta, t \in (t_0 - \tau, t_0), \\ p \in P, t \in (t_0, t_1 - \tau), \text{ singular}, \\ \alpha, t \in (t_1 - \tau, t_1). \end{cases}$$

VIII. Let $\psi_1 < 0$ and $\psi_0 + a\psi_1 = 0$. Then, $\psi_0 + (a - c)\psi_1 > 0$. Thus,

$$p_0(t) = \begin{cases} \beta, t \in (t_0 - \tau, t_1 - \tau), \\ p \in P, t \in (t_1 - \tau, t_1), \text{ singular}. \end{cases}$$

It is clear that, among of the cases I-VIII an optimal control will be such control which transfers the point x_0 into x_1 and minimizes the functional $J(p)$.

Example. Let $D(p) = -p + 1, S(p) = p - 1, \tau = 1, t_0 = 1, t_1 = 3, \alpha = 1, \beta = 2$. Consider the optimization problem

$$\begin{cases} \dot{x}(t) = -p(t) - p(t - 1) + 2, t \in I = [1, 3], p(t) \in P = [1, 2], \\ x(1) = 0, x(3) = -2, \\ J(p) = \int_1^3 p(t) dt \rightarrow \min. \end{cases}$$

The control $p(t) = 3/2, t \in [0, 3]$ is admissible. Indeed,

$$x(3) = \int_1^3 [-p(t) - p(t - 1) + 2] dt = \int_1^3 \left[-\frac{3}{2} - \frac{3}{2} \right] dt + 4 = -2.$$

On the basis of I-VIII we find such control which transfers 0 into -2.

I. $p_0(t) = p \in [1, 2], t \in [0, 1]$ and $p_0(t) = 1, t \in [1, 3]$. We have

$$\begin{aligned} x(3) &= \int_1^3 [-p_0(t) - p_0(t-1)] dt + 4 = 4 - \int_1^3 1 dt - \int_0^2 p_0(t) dt \\ &= 2 - \int_0^1 p dt - \int_1^2 1 dt = 2 - p - 1 = 1 - p. \end{aligned}$$

It is clear that $p_0(t) \in \Omega$ if there exists a number $p \in P$ such that $1 - p = -2$. But the last equality is valid for $p = 3 \notin P$. Thus, $p_0(t) \notin \Omega$.

II. $p_0(t) = 1, t \in [0, 3]$ then

$$x(3) = -4 + 4 = 0,$$

i. e. $p_0(t) \notin \Omega$.

III.

$$p_0(t) = \begin{cases} 2, t \in (0, 1), \\ 1, t \in [1, 3]. \end{cases}$$

We have

$$x(3) = -2 - 2 - 1 + 4 = -1,$$

$p_0(t) \notin \Omega$.

IV.

$$p_0(t) = \begin{cases} 2, t \in (0, 1), \\ 1, t \in (1, 2), \\ 2, t \in (2, 3). \end{cases}$$

$$x(3) = -1 - 2 - 2 - 1 + 4 = -2.$$

$p_0(t) \in \Omega$ and

$$J(p_0) = \int_1^3 p_0(t) dt = \int_1^2 1 dt + \int_2^3 2 dt = 3.$$

V.

$$p_0(t) = \begin{cases} 2, t \in (0, 2), \\ 1, t \in (2, 3). \end{cases}$$

$$x(3) = -2 - 1 - 2 - 1 + 4 = -3$$

$p_0(t) \notin \Omega$.

VI. $p_0(t) = 2, t \in [0, 3]$.

$$x(3) = -8 + 4 = -4$$

$p_0(t) \notin \Omega$.

VII.

$$p_0(t) = \begin{cases} 2, t \in (0, 1), \\ \forall p \in P, t \in (1, 2), \text{ singular} \\ 1, t \in (2, 3). \end{cases}$$

$$x(3) = -p - 1 - 2 - p + 4.$$

$-2p + 1 = -2 \rightarrow p = 3/2$. Thus $p_0(t) \in \Omega$ and $J(p_0) = 2.5$.

VIII.

$$p_0(t) = \begin{cases} 2, t \in (0, 2), \\ \forall p \in P, t \in (2, 3), \text{ singular.} \end{cases}$$

$$x(3) = -2 - p - 4 + 4 = -2 - p.$$

$-2 - p = -2 \rightarrow p = 0$. Thus $p_0(t) \notin \Omega$.

Consequently , the control

$$p_0(t) = \begin{cases} 2, t \in (0, 1), \\ 3/2, t \in (1, 2), \\ 1, t \in (2, 3). \end{cases}$$

is optimal.

4 Conclusion

The scheme provided here allows one to find optimal control for more complex optimization problems and develop a numerical algorithm.

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